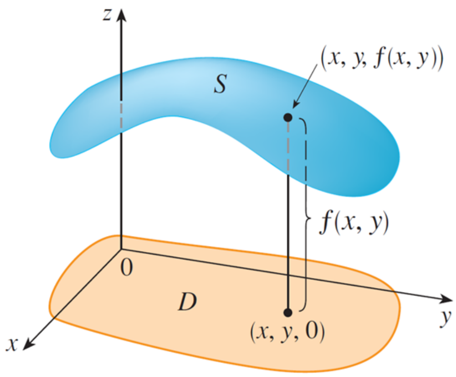
Functions of Several Variables

Definition Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a function defined on D. Then the graph of f on D is a subset in $\mathbb{R}^3$ defined by

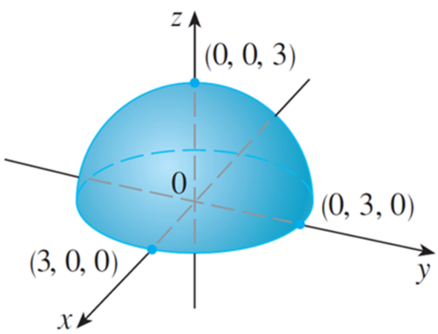
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \text{ and } (x, y) \in D\} \subset \mathbb{R}^3.$$



Examples

- If $(a, b) \neq (0, 0)$, the graph of the linear function $f(x, y) = ax + by + c$ on $\mathbb{R}^2$ is a plane in $\mathbb{R}^3$ given by
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = ax + by + c \text{ and } (x, y) \in \mathbb{R}^2\}.$$
- The graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$ on the closed disk $x^2 + y^2 \leq 9$ is the upper hemisphere with center $(0, 0, 0)$ and radius 3 given by

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = \sqrt{9 - x^2 - y^2} \geq 0 \text{ and } x^2 + y^2 \leq 9\}$$

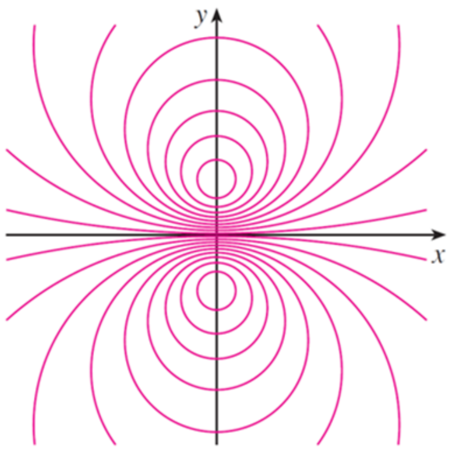


Definition Let D be a subset of $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be a function defined on D. Then a level curve of f at the level k is a subset of D given by

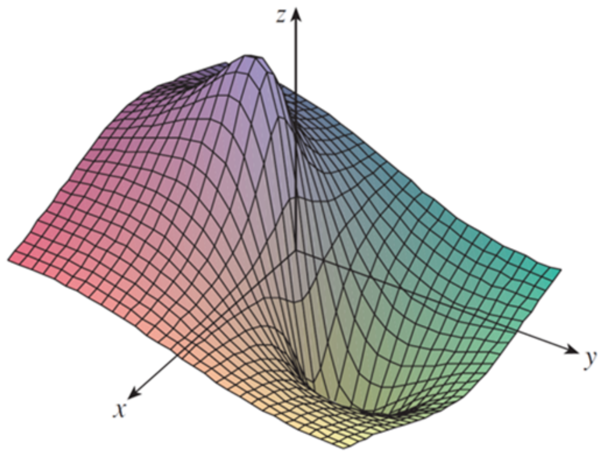
$$L_f(k) = \{(x, y) \in D \mid f(x, y) = k\} \subseteq D.$$

A collection of level curves is called a contour map of f.

Example Let $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$ for $(x, y) \in \mathbb{R}^2$. The following sketches show the level curves and the graph near the origin.



(c) Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$



(d) $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Remark In general, if D is a subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ is a function defined on D. Then the graph of f on D is a subset in $\mathbb{R}^{n+1}$ defined by

$$S = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = f(x_1, \dots, x_n) \text{ and } (x_1, \dots, x_n) \in D\} \subset \mathbb{R}^{n+1}.$$

and a level set (or level curve, surface when $n = 2, 3$ respectively) of f at the level k is a subset of D given by

$$L_f(k) = \{(x_1, \dots, x_n) \in D \mid f(x_1, \dots, x_n) = k\} \subseteq D.$$

Limits and Continuity

Let $p \in \mathbb{R}^n$, $r > 0$ and let $B_r(p)$ denote the ball of center p and radius r defined by

$$B_r(p) = \{x \in \mathbb{R}^n \mid |x - p|^2 = \sum_{i=1}^{n} (x_i - p_i)^2 < r^2\},$$

where $|x - p|$ is the Euclidean distance from x to p.

Definitions Let D be a subset of $\mathbb{R}^n$ and $p \in D$. Then

- p is called an interior point of D if there exists an $r > 0$ such that

$$B_r(p) = \{x \in \mathbb{R}^n \mid |x - p| < r\} \subset D \iff \text{if } x \in B_r(p) \text{ then } x \in D.$$

- p is called a boundary point of D if it is not an interior point of D.
- D is called an open subset of $\mathbb{R}^n$ if every point in D is an interior point of D.

Remark If p is a point in D, then p is either an interior point or a boundary point of D.

Example Let $D = [0, 1] \cup \{2\} \subset \mathbb{R}$. Then $(0, 1)$ is the set of interior points of D while $\{0, 1, 2\}$ is the set of boundary points of D.

Definition Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \in \mathbb{R}$$

if for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\begin{aligned} & \text{if } 0 < |(x,y)-(a,b)| < \delta \text{ then } (x,y) \in D \text{ and } |f(x,y)-L| < \varepsilon \\ \iff & \text{if } (x,y) \in B_\delta((a,b)) \setminus \{(a,b)\} \text{ then } (x,y) \in D \text{ and } |f(x,y)-L| < \varepsilon \end{aligned}$$

Note that if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, then it is unique.

Algebraic Properties of Limits Let f, g be functions of two variables whose domain D includes points arbitrarily close to (a,b). If

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \in \mathbb{R} \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M \in \mathbb{R},$$

then

- **(Sum and Difference Law)** $\lim\limits_{(x,y)\to(a,b)} [f \pm g](x,y) = L \pm M$
- **(Product Law)** $\lim\limits_{(x,y)\to(a,b)} [f \times g](x,y) = L \times M$
- **(Quotient Law)** $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ provided that $g(x,y) \neq 0$ for (x,y) close to (a,b) and the limit of the denominator is not 0.

Proposition (Squeeze Theorem) Let f, ℓ, $r : D \to \mathbb{R}$ be functions of two variables whose domain D includes points arbitrarily close to (a,b). Suppose that

$$\ell(x,y) \leq f(x,y) \leq r(x,y) \quad \text{for all } (x,y) \in D \setminus \{(a,b)\},$$

and

$$\lim_{(x,y)\to(a,b)} \ell(x,y) = L = \lim_{(x,y)\to(a,b)} r(x,y).$$

Then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

Definition Let D be a subset of $\mathbb{R}^2$, $f : D \to \mathbb{R}$ be a function defined on D and let (a,b) be an interior point of D. Then f is said to be continuous at (a,b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b),$$

i.e. for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\begin{aligned} & \text{if } |(x,y)-(a,b)| < \delta \text{ then } (x,y) \in D \text{ and } |f(x,y)-f(a,b)| < \varepsilon \\ \iff & \text{if } (x,y) \in B_\delta((a,b)) \text{ then } (x,y) \in D \text{ and } |f(x,y)-f(a,b)| < \varepsilon \end{aligned}$$

We say that f is continuous on D if f is continuous at every point (a,b) in D.

Algebraic Properties of Continuous Functions Let f and g be functions of two variables whose domain D includes points arbitrarily close to (a,b). If f and g are continuous at (a,b), i.e.

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b) \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = g(a,b),$$

then so is the

- (Sum and Difference) $f \pm g$ since $\lim\limits_{(x,y)\to(a,b)} [f \pm g](x,y) = f(a,b) \pm g(a,b)$.

- (Product) $f \times g$ since $\lim_{(x,y)\to(a,b)} [f \times g](x,y) = f(a,b) \times g(a,b)$.
- (Quotient) $\frac{f}{g}$ provided that $g(a,b) \neq 0$ since $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{f(a,b)}{g(a,b)}$.

Examples

1. Show that $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$.
2. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.
3. Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.
4. Where is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous?

Remark In general, if D is a subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}$ is a real-valued function defined on D includes points arbitrarily close to p, then we say that $\lim_{x\to p} f(x) = L$ if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\begin{aligned} & \text{if } 0 < |x-p| < \delta \text{ then } x \in D \text{ and } |f(x) - L| < \varepsilon, \\ \iff \quad & \text{if } x \in B_\delta(p) \setminus \{p\} \text{ then } x \in D \text{ and } |f(x) - L| < \varepsilon. \end{aligned}$$

If $p \in D$, then we say that f is continuous at p if $\lim_{x\to p} f(x) = f(p)$, i.e. if for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$\begin{aligned} & \text{if } |x-p| < \delta \text{ then } x \in D \text{ and } |f(x) - f(p)| < \varepsilon \\ \iff \quad & \text{if } x \in B_\delta(p) \text{ then } x \in D \text{ and } |f(x) - f(p)| < \varepsilon \end{aligned}$$

Definition Let D be a subset of $\mathbb{R}^2$, $p = (a,b)$ be an interior point of D and let $f : D \to \mathbb{R}$ be a real-valued function defined on D. Then the partial derivative of f with respect to x at (a,b), denoted by $f_x(a,b)$ or $\frac{\partial f}{\partial x}(a,b)$, is defined to be

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \quad \text{provided that the limit exists,}$$

and the partial derivative of f with respect to y at (a,b), denoted by $f_y(a,b)$ or $\frac{\partial f}{\partial y}(a,b)$, is defined to be

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h} \quad \text{provided that the limit exists.}$$

Remark Recall that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x = a$, then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and f is continuous at $x = a$ since

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} \cdot (x-a)\right] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) = 0 \implies \lim_{x\to a} f(x) = f(a).$$

However, the existence of partial derivatives for a function of several variables do not always guarantee the continuity of the function.

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables defined by

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

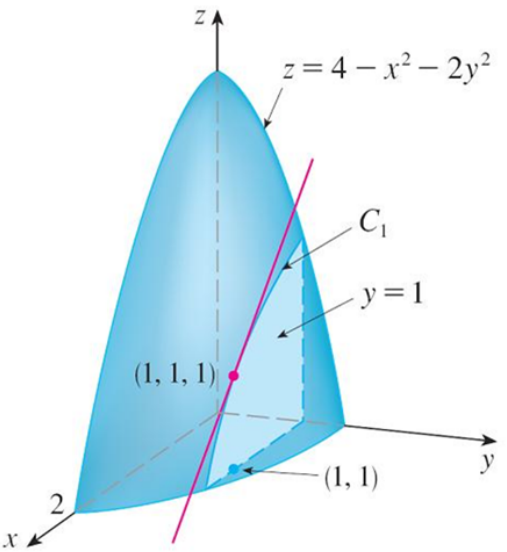
1. Show that f is not continuous at $(0,0)$.
2. Find $f_x(0,0)$ and $f_y(0,0)$.

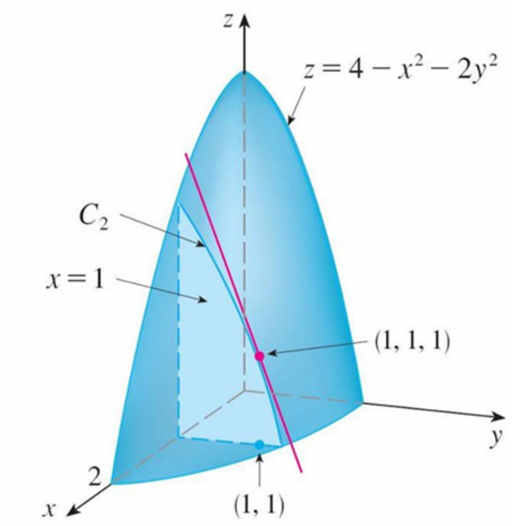
Remark Let D be an open subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a real-valued function defined on D. Suppose that the partial derivatives f_x and f_y exist at every point in D, then the functions f_x, $f_y : D \to \mathbb{R}$ are defined by

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} && \text{differentiate } f \text{ with respect to } x \text{ by treating } y \text{ as a constant,} \\ f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} && \text{differentiate } f \text{ with respect to } y \text{ by treating } x \text{ as a constant.} \end{aligned}$$

Examples

1. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.
2. If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.





3. If $f(x,y,z) = e^{xy}\ln z$, find f_x, f_y, and f_z.
4. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find the second partial derivatives $f_{xx} = (f_x)_x$, $f_{xy} = (f_x)_y$, $f_{yx} = (f_y)_x$, and $f_{yy} = (f_y)_y$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Examples

1. Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function of two variables defined by

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y)\neq(0,0),\\ 0 & \text{if } (x,y)=(0,0).\end{cases}$$

Show that

$$f_x(x,y)=\begin{cases}\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \quad \text{by direct differentiation},\\ 0 & \text{if } (x,y)=(0,0) \quad \text{by definition of partial derivative},\end{cases}$$

$$f_y(x,y)=\begin{cases}\dfrac{-y^4x-4y^2x^3+x^5}{(y^2+x^2)^2} & \text{if } (x,y)\neq(0,0) \quad \text{by direct differentiation},\\ 0 & \text{if } (x,y)=(0,0) \quad \text{by definition of partial derivative},\end{cases}$$

and that $f_{xy}(0,0)=-1\neq 1=f_{yx}(0,0)$ by definition of partial derivatives.

2. Show that the function $u(x,y)=e^x\sin y$ is a solution of the Laplace equation, that is $\Delta u=u_{xx}+u_{yy}=0$.

Definition Let $D\subset\mathbb{R}^2$, (a,b) be an interior point of D and let $f:D\to\mathbb{R}$ be a function defined on D. Then f is called differentiable at (a,b) if there exists $(\ell_1,\ell_2)\in\mathbb{R}^2$ such that

$$\lim_{(x,y)\to(a,b)}\frac{|f(x,y)-f(a,b)-\ell_1(x-a)-\ell_2(y-b)|}{|(x,y)-(a,b)|}=0.$$

$$\iff \lim_{(x,y)\to(a,b)}\frac{|\varepsilon(x,y)|}{\sqrt{(x-a)^2+(y-b)^2}}=0,\ \text{where } \varepsilon(x,y)=f(x,y)-f(a,b)-\ell_1(x-a)-\ell_2(y-b)$$

Theorem If f is differentiable at (a,b), then the partial derivatives f_x and f_y exist at (a,b) and $(\ell_1,\ell_2)=(f_x(a,b),f_y(a,b))$.

Proof Since

$$0=\lim_{(x,b)\to(a,b)}\frac{|f(x,b)-f(a,b)-\ell_1(x-a)-\ell_2(b-b)|}{|(x,b)-(a,b)|}=\lim_{x\to a}\frac{|f(x,b)-f(a,b)-\ell_1(x-a)|}{|x-a|}$$

$$\implies 0=\lim_{x\to a}\frac{f(x,b)-f(a,b)-\ell_1(x-a)}{x-a}=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}-\ell_1=f_x(a,b)-\ell_1,$$

and

$$0=\lim_{(a,y)\to(a,b)}\frac{|f(a,y)-f(a,b)-\ell_1(a-a)-\ell_2(y-b)|}{|(a,y)-(a,b)|}=\lim_{y\to b}\frac{|f(a,y)-f(a,b)-\ell_2(y-b)|}{|y-b|}$$

$$\implies 0=\lim_{y\to b}\frac{f(a,y)-f(a,b)-\ell_2(y-b)}{y-b}=\lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}-\ell_2=f_y(a,b)-\ell_2.$$

Theorem If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b), that is,

$$\lim_{(x,y)\to(a,b)}\frac{|f(x,y)-f(a,b)-f_x(a,b)(x-a)-f_y(a,b)(y-b)|}{|(x,y)-(a,b)|}=0.$$

Proof For each $\varepsilon > 0$, since f_x and f_y exist near (a,b) and are continuous at $p = (a,b)$, there exists a $\delta > 0$ such that if $(x,y) \in B_\delta(p)$, then

$$|f_x(x,y) - f_x(a,b)| + |f_y(x,y) - f_y(a,b)| < \varepsilon.$$

Let $g : B_\delta(p) \to \mathbb{R}$ be defined by

$$g(x,y) = f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b) \quad \text{for } (x,y) \in B_\delta(p).$$

Then $g(a,b) = 0$,

$$g_x(x,y) = f_x(x,y) - f_x(a,b) \quad \text{and} \quad g_y(x,y) = f_x(x,y) - f_y(a,b).$$

Let the line segment in $B_\delta(p)$ from (x,y) to $p = (a,b)$ be given by

$$r(t) = (a,b) + t(x-a, y-b),\ t \in [0,1],$$

and consider the function

$$g(r(t)) = g(x(t), y(t)) = g(a + t(x-a), b + t(y-b)) \quad \text{for } t \in [0,1].$$

By the Mean Value Theorem, there is a $0 < t_0 < 1$ with $r(t_0) = (x_0, y_0)$, such that

$$\begin{aligned}
|g(r(1)) - g(r(0))| &= |\frac{d}{dt}g(r(t))|_{t=t_0}(1-0)| = |\frac{d}{dt}g(x(t),y(t))|_{t=t_0}| \\
&= |g_x(x_0,y_0)x'(t_0) + g_y(x_0,y_0)y'(t_0)| \\
&= |[f_x(x_0,y_0) - f_x(a,b)](x-a) + [f_y(x_0,y_0) - f_y(a,b)](y-b)| \\
&\le (|f_x(x_0,y_0) - f_x(a,b)| + |f_y(x_0,y_0) - f_y(a,b)|)\sqrt{(x-a)^2 + (y-b)^2} \\
&< \varepsilon\sqrt{(x-a)^2 + (y-b)^2} = \varepsilon|(x,y) - (a,b)|
\end{aligned}$$

Since

$$g(r(1)) = g(x,y) = f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b),\ g(r(0)) = g(a,b) = 0,$$

we have

$$|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)| = |g(r(1)) - g(r(0))| < \varepsilon|(x,y) - (a,b)|,$$

which implies that

$$\frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{|(x,y) - (a,b)|} < \varepsilon.$$

Since $\varepsilon > 0$ is an arbitrary positive number, we have

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{|(x,y) - (a,b)|} = 0$$

and f is differentiable at (a,b).

Equation of a Tangent Plane Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a function with continuous partial derivatives. Then the plane tangent to the surface

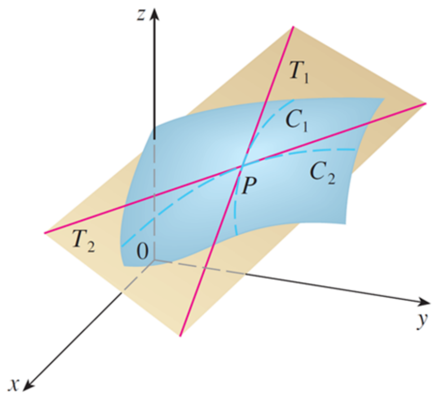
$$S = \{(x, y, z) \in \mathbb{R}^3 \mid z = f(x, y) \text{ and } (x, y) \in D\},$$

at the point $P(x_0, y_0, z_0) \in S$ has an equation

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0).$$

Proof Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Then the plane tangent to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2.



Since $C_1 = S \cap \{(x, y_0, z) \mid (x, z) \in \mathbb{R}^2\}$ and $C_2 = S \cap \{(x_0, y, z) \mid (y, z) \in \mathbb{R}^2\}$ are curves in S, we may parametrize C_1 and C_2 by vector function

$$\begin{aligned} C_1 : \quad & r_1(x) = (x, y_0, z) \overset{C_1 \subseteq S}{=} (x, y_0, f(x, y_0)) \quad \text{for } (x, y_0) \in D \implies r_1'(x_0) = (1, 0, f_x(x_0, y_0)) \,/\!/\, T_1 \\ C_2 : \quad & r_2(y) = (x_0, y, z) \overset{C_2 \subseteq S}{=} (x_0, y, f(x_0, y)) \quad \text{for } (x_0, y) \in D \implies r_2'(y_0) = (0, 1, f_y(x_0, y_0)) \,/\!/\, T_2 \end{aligned}$$

This implies that the tangent plane to the surface S at the point P is perpendicular the vector

$$(1, 0, f_x(x_0, y_0)) \times (0, 1, f_y(x_0, y_0)) = (-f_x(x_0, y_0), -f_y(x_0, y_0), 1),$$

and has an equation

$$\begin{aligned} & (x - x_0, y - y_0, z - z_0) \cdot (-f_x(x_0, y_0), -f_y(x_0, y_0), 1)\rangle = 0 \quad \text{since } \cos\frac{\pi}{2} = 0 \\ \iff \quad & -f_x(x_0, y_0)\,(x - x_0) - f_y(x_0, y_0)\,(y - y_0) + (z - z_0) = 0 \\ \iff \quad & z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0). \end{aligned}$$

Example Find an equation for the plane tangent to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Definition Let $D \subset \mathbb{R}^2$, (a, b) be an interior point of D and let $f : D \to \mathbb{R}$ be a function with continuous partial derivatives. The linear function $L : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$L(x, y) = f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b) \quad \text{for all } (x, y) \in \mathbb{R}^2$$

is called the linearization of f at (a, b) and the approximation

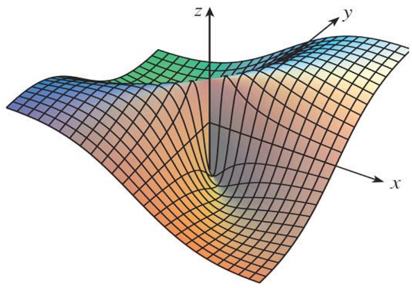
$$f(x, y) \approx f(a, b) + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b)$$

is called the linear approximation or the tangent plane approximation of f at (a, b).

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

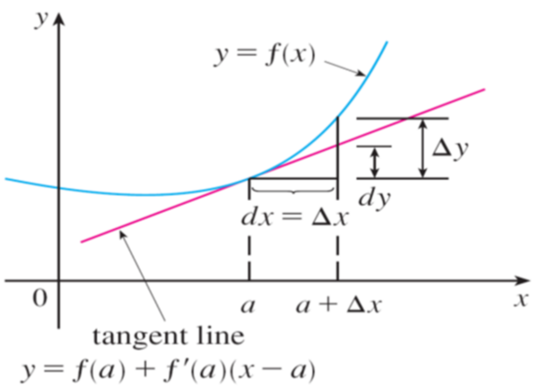
Note that $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$, but f_x and f_y are not continuous at $(0, 0)$, and the surface $z = f(x, y)$ does not have a tangent plane at $(0, 0)$.

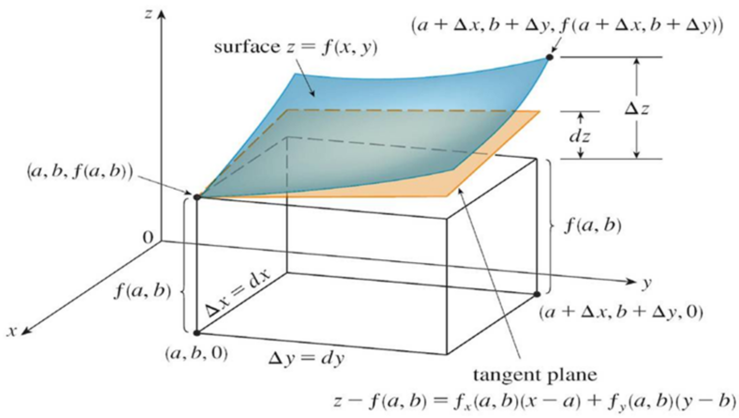


Example Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Definition Let D be an open subset of $\mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a differentiable function defined on D. The differential df is defined by

$$df = f_x(x, y)dx + f_y(x, y)dy$$





Note that the differentials

- $dy = f'(x)dx =$ the change in height of the tangent line,
- $dz = f_x(x, y)dx + f_y(x, y)dy =$ the change in height of the tangent plane,

whereas the increments

- $\Delta y = f(x + \Delta x) - f(x) =$ the change in height of the curve $y = f(x)$,

- $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) =$ the change in height of the surface $z = f(x, y)$,

and $\Delta z - dz = R =$ the gaps between surface and tangent plane satisfies that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{R}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0 \quad \text{by Taylor's Theorem.}$$

Examples

1. If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential $dz = df$.

2. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

3. The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within ε cm.

- Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements. [Let x, y and z be the dimensions of the box. Since its volume $V = xyz$ and the error $\Delta V \approx dV = yzdx + xzdy + xydz = 9900\varepsilon$, the maximum error in the calculated volume is about 9900 times larger than the error in each measurement taken.]
- What is the estimated maximum error in the calculated volume if the measured dimensions are correct to within 0.2 cm. [If the largest error in each measurement is $\varepsilon = 0.2$ cm, then $dV = 9900(0.2) = 1980$, so an error of only 0.2 cm in measuring each dimension could lead to an error of approximately $1980\,\text{cm}^3$ (1.1%) in the calculated volume $180,000\,\text{cm}^3$.]

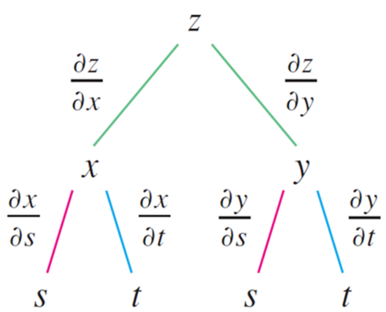
Chain Rule

(a) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

(b) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$



(c) In general, if u is a differentiable function of n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. The u is a function of $t_1, t_2, \ldots, t_m$ and for each $i = 1, 2, \ldots, m$, we have

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

Examples

1. If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\dfrac{dz}{dt}$ when $t = 0$.
2. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.
3. If F is differentiable on a disk containing (a, b), then the equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x near the point (a, b) and we can apply the Chain Rule to differentiate both sides of $F(x, y) = 0$ with respect to x, and obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y} \quad \text{if } \frac{\partial F}{\partial y} \neq 0,$$

   e.g. find y' if $x^3 + y^3 = 6xy$.
4. Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$, i.e. $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f.
   If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to x and y, and obtain

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \overset{\partial y/\partial x = 0}{\underset{\partial x/\partial x = 1}{\Longrightarrow}} \frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0,$$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 \overset{\partial x/\partial y = 0}{\underset{\partial y/\partial y = 1}{\Longrightarrow}} \frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z} \quad \text{if } \frac{\partial F}{\partial z} \neq 0.$$

Directional Derivatives and the Gradient Vector

Definition Let D be a subset of $\mathbb{R}^2$, (x_0, y_0) be an interior point of D, and let $f : D \to \mathbb{R}$ be a function on D. Then the directional derivative of f at (x_0, y_0) in the direction of a unit vector $u = (a, b)$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad \text{if this limit exists.}$$

Remarks

(a) If $u = \mathbf{i} = (1, 0)$, then $D_{\mathbf{i}} f = f_x$ and if $u = \mathbf{j} = (0, 1)$, then $D_{\mathbf{j}} f = f_y$.

(b) If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $u = (a, b)$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b = (f_x(x, y), f_y(x, y)) \cdot (a, b) = \nabla f(x, y) \cdot (a, b),$$

where $\nabla f(x, y) = (f_x(x, y), f_y(x, y))$ is called the gradient (vector) of f, or grad f, at (x, y).

Furthermore, since $u = (a, b)$ is a unit vector, there exists an angle θ measured from the positive x-axis to u in the counterclockwise direction such that $u = (\cos\theta, \sin\theta)$. Then

$$D_u f(x, y) = \nabla f(x, y) \cdot (\cos\theta, \sin\theta) \quad \text{is a function of } x,\ y \text{ and } \theta.$$

Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_u f(p)$ is $|\nabla f(p)|$ and it occurs when u has the same direction as the gradient vector $\nabla f(p)$.

Examples

1. Let $f(x, y) = \sin x + e^{xy}$, $(x, y) \in \mathbb{R}^2$. Find $\nabla f(1, 0)$ and find points (x, y) such that $\nabla f(x, y) = (0, 0)$.
2. Let $f(x, y, z) = x \sin yz$, $(x, y, z) \in \mathbb{R}^3$. (a) Find the gradient of f and (b) find the directional derivative of f at $(1, 3, 0)$ in the direction of $v = \mathbf{i} + 2\mathbf{j} - \mathbf{k} = (1, 2, -1)$.
3. Let $f(x, y) = xe^y$, $(x, y) \in \mathbb{R}^2$. (a) Find the rate of change of f at the point $p = (2, 0)$ in the direction from p to $q = (1/2, 2)$, and (b) determine the direction in which f has the maximum rate of change and (c) find the maximum rate of change of f at p.

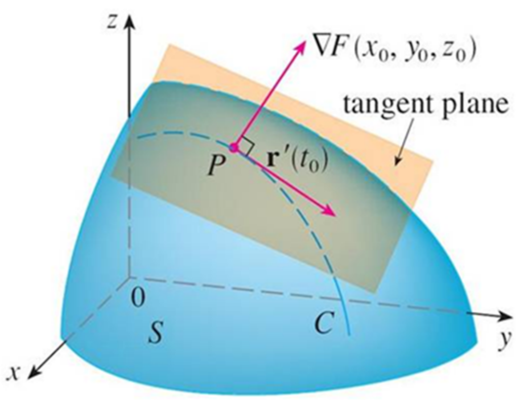
Tangent Planes to Level Surfaces

Suppose that

- $S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = k\}$ is a level surface of F in $\mathbb{R}3$,
- $p = (x_0, y_0, z_0)$ is a point in S,
- $C \subset S$ is any differentiable curve in S passing through p, and parametrized by $r(t) = (x(t), y(t), z(t))$, $t \in I = (a, b)$, with $r(t_0) = p$ for some $t_0 \in I$.

Since $C = \{r(t) \mid t \in I\} \subset S$, and by the Chain Rule, we have

$$F(x(t), y(t), z(t)) = k \implies \frac{d}{dt}F(x(t), y(t), z(t)) = 0 \quad \text{for all } t \in I$$
$$\implies \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial x}) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) = \nabla F(r(t)) \cdot r'(t) = 0 \quad \text{for all } t \in I.$$



In particular, we have

$$\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0,$$

which implies that if $\nabla F(x_0, y_0, z_0) \neq (0,0,0)$, $\nabla F(x_0, y_0, z_0)$ is perpendicular to the tangent vector $r'(t_0)$ to any curve C in S passing through $p = r(t_0) = (x_0, y_0, z_0)$, and the tangent plane to the level surface $F(x, y, z) = k$ at $p = (x_0, y_0, z_0)$ has an equation

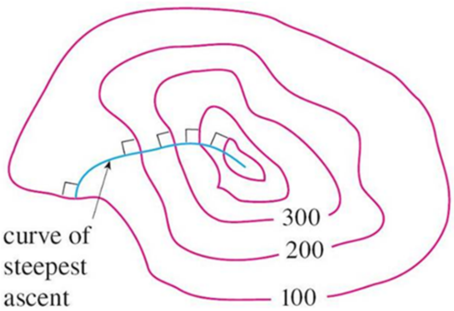
$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

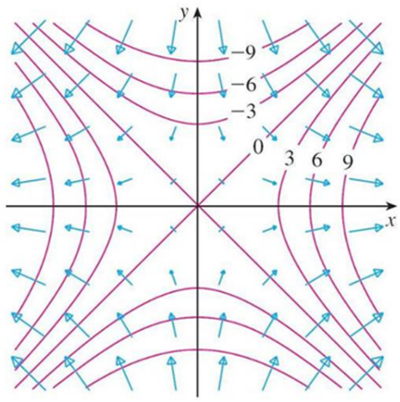
Example Find the equations of the tangent plane and normal line to ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \quad \text{at the point } (-2, 1, -3).$$

Properties of the Gradient Vector Let f be a differentiable function of two or three variables and suppose that $\nabla f(p) \neq \mathbf{0}$ (zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$). Then

- The directional derivative of f at p in the direction of a unit vector u is given by $D_u f(p) = \nabla f(p) \cdot u$.
- $\nabla f(p)$ points in the direction of maximum rate of increase of f at p, and that maximum rate of change is $|\nabla f(p)|$.
- $\nabla f(p)$ is perpendicular to the level curve or level surface of f through p.





Examples The figure on the right shows level sets of a height function or $f(x, y) = x^2 - y^2$ with a gradient vector fields.

Definition Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a real-valued function defined on D. Then

- f is said to have a local maximum value at p if there exists $r > 0$ such that $B_r(p) \subset D$ and

$$f(x) \leq f(p) \quad \text{for all } x \in B_r(p).$$

- f is said to have a local minimum value at p if there exists $r > 0$ such that $B_r(p) \subset D$ and

$$f(x) \geq f(p) \quad \text{for all } x \in B_r(p).$$

Therem (First derivatives Test) If f has a local maximum or minimum at p and if the first partial derivatives of f exist at p, then $\nabla f(p) = (f_{x_1}, f_{x_2}, \ldots, f_{x_n})(p) = (0, 0, \ldots, 0) = \mathbf{0} \in \mathbb{R}^n$.

Definition A point $p \in D$ is called a critical point (or stationary point) of f if either $\nabla f(p) = \mathbf{0} \in \mathbb{R}^n$ or if $\nabla f(p)$ does not exist.

Example Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$, $(x,y) \in \mathbb{R}^2$. Find the critical points and extreme values (or critical values) of f if exist.

Classification of Extreme Values Theorem (Second Derivatives Test) Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a function defined on D, p be an interior point of D and let $B_r(p) \subset D$ be an open disk in D. Suppose that the second partial derivatives of f are continuous on $B_r(p)$, $\nabla f(p) = (f_x(p), f_y(p)) = (0,0)$ and that

$$D = f_{xx}(p)f_{yy}(p) - [f_{xy}(p)]^2 = \begin{vmatrix} f_{xx}(p) & f_{xy}(p) \\ f_{yx}(p) & f_{yy}(p) \end{vmatrix}.$$

(a) If $D > 0$ and $f_{xx}(p) > 0$, then $f(p)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(p) < 0$, then $f(p)$ is a local maximum.

(c) If $D < 0$, then $(p, f(p))$ is a saddle point of the graph of f.

(d) If $D = 0$, the test is inconclusive: f could have a local maximum or local minimum at p or $(p, f(p))$ could be a saddle point of the graph of f.

Outline of the Proof Let $u = (h,k)$ be a unit vector. For any $|t| < r$, since $f(p+tu)$ has continuous second derivative for each $t \in (-r, r)$ and since

$$\frac{d}{dt} f(p+tu)|_{t=0} = \big[f_x(p+tu)h + f_y(p+tu)k\big]|_{t=0} = f_x(p)h + f_y(p)k = \nabla f(p) \cdot (h,k) = 0,$$

and

$$\begin{aligned} &\frac{d^2}{dt^2} f(p+tu)|_{t=0} = \frac{d}{dt}\big[f_x(p+tu)h + f_y(p+tu)k\big]|_{t=0} \\ =\ & \big[f_{xx}(p+tu)h^2 + f_{xy}(p+tu)hk + f_{yx}(p+tu)kh + f_{yy}(p+tu)k^2\big]|_{t=0} \\ =\ & f_{xx}(p)h^2 + 2f_{xy}(p)hk + f_{yy}(p)k^2 = f_{xx}(p)\left(h + \frac{f_{xy}(p)}{f_{xx}(p)}k\right)^2 + \frac{k^2}{f_{xx}(p)}\big(f_{xx}(p)f_{yy}(p) - f_{xy}^2(p)\big), \end{aligned}$$

so by the Taylor's Theorem and for each $|t| < r$ and for any unit vector $u = (h,k) \in \mathbb{R}^2$, we have

$$\begin{aligned} f(p+tu) - f(p) &= \left(\frac{d}{dt} f(p+tu)|_{t=0}\right) t + \left(\frac{1}{2}\frac{d^2}{dt^2} f(p+tu)|_{t=0}\right) t^2 + R(t) \\ &= \left[f_{xx}(p)\left(h + \frac{f_{xy}(p)}{f_{xx}(p)}k\right)^2 + \frac{k^2}{f_{xx}(p)}\big(f_{xx}(p)f_{yy}(p) - f_{xy}^2(p)\big)\right]\frac{t^2}{2} + R(t), \end{aligned}$$

where $\lim_{t\to 0} \frac{R(t)}{t^2} = 0$. Hence, the theorem follows by using the second derivative test for functions of one variable.

Remark Setting $a = f_{xx}(p)$, $b = f_{xy}(p)$, $c = f_{yy}(p)$, note that

- if $a \neq 0$ and $ac - b^2 > 0$, then

$$\begin{aligned} ax^2 + 2bxy + cy^2 &= a\left(x^2 + \frac{2b}{a}xy + \frac{b^2}{a^2}y^2\right) + \left(c - \frac{b^2}{a}\right)y^2 \\ &= a\left(x + \frac{b}{a}y\right)^2 + \frac{ac-b^2}{a}y^2 \begin{cases} \geq 0 & \text{if } a > 0 \\ \leq 0 & \text{if } a < 0 \end{cases} \end{aligned}$$

- if $a \neq 0$ and $ac - b^2 < 0$, then

$$\begin{aligned} ax^2 + 2bxy + cy^2 &= a\left(x + \frac{b}{a}y\right)^2 - \frac{b^2 - ac}{a}y^2 \\ &= a\left(x + \frac{b}{a}y + \frac{\sqrt{b^2 - ac}}{a}y\right)\left(x + \frac{b}{a}y - \frac{\sqrt{b^2 - ac}}{a}y\right) \end{aligned}$$

  and $(0,0,0)$ is a saddle point of the graph of $z = ax^2 + 2bxy + cy^2$ since $x + \frac{b \pm \sqrt{b^2 - ac}}{a}y = 0$ are distinct lines dividing xy-plane into 4 regions around $(0,0)$.

- if $a = 0$ and $ac - b^2 < 0 \implies b \neq 0$, then $ax^2 + 2bxy + cy^2 = by\,(2x + cy)$ and $(0,0,0)$ is a saddle point of the graph of $z = ax^2 + 2bxy + cy^2$ since $y = 0$ and $2x + cy = 0$ are distinct.

Definitions Let D be a subset of $\mathbb{R}^n$ and let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on D. Then

- D is called a bounded subset of $\mathbb{R}^n$ if there exists a rectangular box $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq n\}$ such that $D \subset R$, or if there exists $r > 0$ such that $D \subset B_r(\mathbf{0})$, where $\mathbf{0} \in \mathbb{R}^n$.
- D is called an open subset of $\mathbb{R}^n$ if for each $p \in D$ there exists $r > 0$ such that $B_r(p) \subset D$, i.e. every point in D is an interior point of D.
- D is called a closed subset of $\mathbb{R}^n$ if its complement $D^c = \{x \in \mathbb{R}^n \mid x \notin D\}$ is an open subset of $\mathbb{R}^n$.
- $f(p)$ is called the absolute maximum (value) of f on D if $f(x) \leq f(p)$ for all $x \in D$; $f(p)$ is called the absolute minimum (value) of f on D if $f(x) \geq f(p)$ for all $x \in D$.

Theorem (Extreme Value Theorem) If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous on a closed, bounded set D in $\mathbb{R}^n$, then there exist p, $q \in D$ such that

$$f(p) \geq f(x) \geq f(q) \quad \text{for all } x \in D \iff \max_{x \in D} f(x) = f(p) \text{ and } \min_{x \in D} f(x) = f(q).$$

Remark If f has extreme values at p, $q \in D \subset \mathbb{R}^2$, since p (or q) is either a critical point of f or a boundary point D, we shall find the absolute maximum and minimum of f on D as follows.

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example Find the absolute maximum and minimum of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

Method of Lagrange Multipliers

Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function defined on $\mathbb{R}^3$ and let $S = \{(x, y, z) \mid g(x, y, z) = k\}$ be a (level) surface defined by $g(x, y, z) = k$. Suppose that

- $\nabla g \neq \mathbf{0}$ (vector) on the surface $g(x, y, z) = k$,
- there is a point $p = (x_0, y_0, z_0) \in S$ such that

$$\text{either } f(p) = \max\{f(x, y, z) \mid g(x, y, z) = k\} \text{ or } f(p) = \min\{f(x, y, z) \mid g(x, y, z) = k\}.$$

Let C be a smooth curve passing through p on S given by the vector equation

$$C: \quad r(t) = (x(t), y(t), z(t)),\ t \in I = (a, b) \text{ and } r(t_0) = p \text{ for some } t_0 \in I.$$

Since $f(r(t))$ has an extreme value at an interior point $t = t_0$ and $g(x(t), y(t), z(t)) = k$ for all $t \in I$, we have

$$\begin{aligned}
& 0 = \frac{d}{dt} f(x(t), y(t), z(t))|_{t=t_0} = \nabla f(p) \cdot r'(t_0), \\
& 0 = \frac{d}{dt} k = \frac{d}{dt} g(x(t), y(t), z(t))|_{t=t_0} = \nabla g(p) \cdot r'(t_0), \\
\Longrightarrow\ & \nabla f(p) \perp r'(t_0),\ \nabla g(p) \perp r'(t_0) \text{ for each } r'(t_0) \neq \mathbf{0} \in T_{r(t_0)} S \\
\Longrightarrow\ & \nabla f(p),\ \nabla g(p) \perp T_p S \subset \mathbb{R}^3 \text{ and } \nabla f(p) \mathbin{/\!/} \nabla g(p).
\end{aligned}$$

This suggests that we can use the following procedures (Method of Lagrange Multipliers) to find the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Step 1. Find all values of x, y, z and λ such that

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) & \text{(3 equations of } x,\ y,\ z,\ \lambda), \\ g(x, y, z) = k & \text{(an equation of } x,\ y,\ z). \end{cases}$$

Step 2. Evaluate f at all the points (x, y, z) that result from **Step** 1. The largest of these values is the maximum value of f and the smallest is the minimum value of f.

Example A rectangular box without a lid is to be made from 12m^2 of cardboard. Find the maximum volume of such a box (with x, y, z being the length, width and height, respectively).

Solution To maximize $V = xyz$ subject to $A = xy + 2xz + 2yz = 12$, we find all possible x, y, z, λ such that

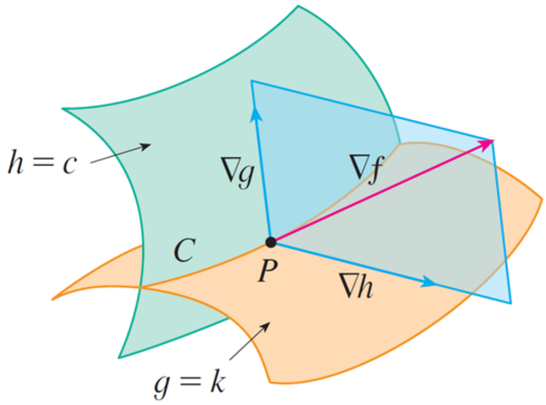
$$\begin{aligned}
& (V_x, V_y, V_z) = \lambda(A_x, A_y, A_z),\ A = 12 \\
\Longleftrightarrow\ & yz \overset{(1)}{=} \lambda(y + 2z),\ xz \overset{(2)}{=} \lambda(x + 2z),\ xy \overset{(3)}{=} \lambda(2x + 2y),\ xy + 2xz + 2yz \overset{(4)}{=} 12 \\
\underset{x(1)-z(3)}{\overset{x(1)-y(2)}{\Longleftrightarrow}}\ & 2\lambda(x - y)z = 0,\ xz \overset{(2)}{=} \lambda(x + 2z),\ \lambda(y - 2z)x = 0,\ xy + 2xz + 2yz = 12 \\
\Longrightarrow\ & x = y = 2z,\ xz \overset{(2)}{=} \lambda(x + 2z),\ xy + 2xz + 2yz = 12 \\
\Longrightarrow\ & 2z^2 \overset{(2)}{=} 4\lambda z,\ xy + 2xz + 2yz = 12 \\
\Longrightarrow\ & z \overset{(2)}{=} 2\lambda,\ x = y = 2z = 4\lambda,\ xy + 2xz + 2yz = 48\lambda^2 = 12
\end{aligned}$$

Thus we have $\lambda = \dfrac{1}{2}$, $x = y = 2$, $z = 1$, and the maximum volume $V = V(2, 2, 1) = 4$.

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$.

Following the method of Lagrange multiplier, we need to find all values of x, y, z, λ and μ such that

$$\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z), \\ g(x, y, z) = k, \\ h(x, y, z) = c. \end{cases}$$



Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$.

Example Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution To maximize $f(x, y, z) = x + 2y + 3z$ subject to $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$, we find all possible x, y, z, λ, μ such that

$$\begin{cases} (f_x, f_y, f_z) = \lambda(g_x, g_y, g_z) + \mu(h_x, h_y, z_z), \\ g = 1, \\ h = 1 \end{cases}$$

$$\iff \begin{cases} (f_x, f_y, f_z) = \lambda(g_x, g_y, g_z) + \mu(h_x, h_y, z_z), \\ g = 1, \\ h = 1 \end{cases}$$

$$\iff \begin{cases} (1, 2, \underline{3}) = \lambda(1, -1, 1) + \mu(2x, 2y, 0) = (\lambda + 2\mu x, -\lambda + 2\mu y, \underline{\lambda}), \\ x - y + z = 1, \\ x^2 + y^2 = 1 \end{cases}$$

$$\iff \begin{cases} \lambda = 3,\ 1 \stackrel{(1)}{=} 3 + 2\mu x,\ 2 \stackrel{(2)}{=} -3 + 2\mu y, \\ x - y + z = 1, \\ x^2 + y^2 = 1 \end{cases}$$

$$\stackrel{x(2)-y(1)}{\Longrightarrow} \quad \lambda = 3,\ x \stackrel{(3)}{=} -\frac{2}{5}y,\ x - y + z \stackrel{(4)}{=} 1,\ x^2 + y^2 \stackrel{(5)}{=} 1$$

$$\stackrel{(3)\to(5)}{\Longrightarrow} \quad \lambda = 3, y = \pm\frac{5}{\sqrt{29}},\ x \stackrel{(3)}{=} \mp\frac{2}{\sqrt{29}},\ x - y + z \stackrel{(4)}{=} 1$$

$$\stackrel{(4)}{\Longrightarrow} \quad \lambda = 3, y = \pm\frac{5}{\sqrt{29}},\ x \stackrel{(3)}{=} \mp\frac{2}{\sqrt{29}},\ z \stackrel{(4)}{=} 1 \pm \frac{7}{\sqrt{29}}$$

Hence $f(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}},) = 3 + \sqrt{29}$ and $f(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}},) = 3 - \sqrt{29}$ are respectively the maximum and minimum values of f subject to $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$.